Interpolated Surfaces using Delaunay Triangulation

Key Concept
Delaunay triangulation can be used to create a continuous surface (“terrain”) from a set of data points by creating a triangular mesh or surface of triangular planes connecting the data points. Delaunay triangulation is considered to be a desirable approach for creating natural-looking surfaces because minimum interior angles of all triangles are maximized and triangles are as equiangular as possible, thus avoiding long, thin triangles.

Delaunay triangles are constructed in such a way that a circumscribed circle of the triangles contains no interior points. The Delaunay circumscribed circle, or circumcircle, is a circle which passes through all the vertices of the triangle as illustrated in figure 1.

![Figure 1](image1.png)

**Figure 1.** The triangle in the image on the left meets the criteria for a Delaunay triangle because it satisfies the “empty circle” property, i.e., a circumcircle of the triangles vertices does not contain any other data points. The triangle in the right-hand image is “illegal” because its circumcircle contains other data points.

As shown in Figure 1, there is more than one way to construct triangles from a set of data points. Figure 2 is a triangular mesh constructed over a set of data points. The right-hand image illustrates that the selected triplet of data points AEC is not a Delaunay triangle because its circumcircle contains an interior point.

![Figure 2](image2.png)

**Figure 2.** One possible construction of a triangular mesh with data points as vertices of the triangles. A circumcircle of triangle AEC reveals an interior data point, D.

Flipping the common edge AC for the common edge DE produces a Delaunay triangulation as shown in Figure 3.
Figure 3. Triangulation of the same data points as where shown in Figure 2. The common edge AE has been flipped to AC to produce a new configuration of triangles. The image on the right illustrates the circumcircles for all the triangles. This triangular mesh is a Delaunay triangulation because there are no interior data points in any of the circumcircles.

Interpolation to a regular grid
The Delaunay triangulation uses data points to creates a continuous surface of triangular planes that can then be sampled into any regular grid (Figure 4) regardless of the grid cell size or grid placement.

Figure 4. A regular grid overlain on a Delaunay surface to produce a raster file of values.

Delaunay triangulation from a regular grid
One of the applications for which this supporting documentation has been prepared is the interpolation of hydrologic model water stage outputs to a finer resolution. The South Florida Water Management Model (SFWMM), for example, has a spatial resolution of 2 miles x 2 miles on a regular grid. To approximate a finer resolution, the centroid of each grid cell is assigned the water stage value of the cell and the centroids are used as data points in the Delaunay triangulation.

Figure 5. Triangulation of a regular grid. Centroid data points are green dots in the center of each of the original grid cells shown in grey. The resulting Delaunay triangulation edges and triangle planes are in shades of blue.
Extra Credit: Delaunay triangulation and the Voronoi diagram

Given a set of data points on a plane, a Voronoi diagram segments the plane into polygons such that all the locations in any particular polygon are closer to the one data point within the polygon than to any other data point. Figure 6 is an example of scattered data points showing each Voronoi polygon in a different color.

![Figure 6. Voronoi diagram.](image1)

![Figure 7. Data points are green dots, Delaunay triangles are shown in blue and Voronoi polygons are yellow lines. The white circles are the Delaunay circumcircles.](image2)

Delaunay triangulation and Voronoi diagrams are closely linked. Data points are the vertices of the Delaunay triangle. As can be seen in Figure 7, the centroids of Delaunay circumcircles are the vertices of Voronoi polygons. Because of this relationship, Voronoi diagrams can easily be created from Delaunay triangles and vis versa.

Selected references for additional information


Prepared by Leonard Pearlstine, Everglades National Park         April 23, 2010